

Depth in Bingo Closure

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Abstract

Bingo is played on a 5×5 grid. Take the 25 squares to be the ground set of a closure system in which square s is *dependent* on a set S of squares iff s completes a *line* — a row, column, or diagonal — with squares that are already in S . The closure of a set S is obtained via an iterative process in which, at each stage, the squares dependent upon the current state are added. In this paper we establish for the $n \times n$ Bingo board the maximum number of steps required in this closure process.

1 Introduction

If you are playing a game of Bingo and you find that you already have 4 squares on any row, column, or diagonal, then the fifth square on that line takes of special interest. If you get that fifth square, it *completes* a line and you win. Convexity, and more generally closure, is based on the idea of filling gaps, closing holes, completing some set. A common and convenient way to express this is in terms on of *dependency*. Here linear algebra is a model: a subspace is a set of vectors which contains all vectors dependent on it.

Bingo closure, of course, can be considered on any $n \times n$ grid. A square s is *dependent* on a set S of squares iff s completes a *line* — a row, column, or diagonal — with squares that are already in S . Let X denote the n^2 squares on an $n \times n$ grid. Dependency defines a set map $\varphi : \text{Pow}(X) \rightarrow \text{Pow}(X)$ where $\varphi(S)$ is the set of all squares dependent on S .

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●		●		●
●	○	a	○	●
○		●		○
●	●	○	●	●
●		b		●

Board 1

2	●	1	●	●
9	●	●	8	●
10	●	●	9	●
●	5	●	●	4
7	6	●	●	3

Board 2

Figure 1: Bingo Closure.

For example in Board 1 in Figure 1, let S be the set of 12 squares marked by solid black dots. Each of the 5 squares marked by open circles completes a line with squares in S . Thus $\varphi(S)$ is the set A of squares marked by open circles. Our definition of dependency has the undesirable peculiarity that a point *in* a set may fail to be dependent on that set. In particular it is the case in Board 1 that no square in S is dependent on S . However, if we look at $S \cup A$ both diagonals belong to $S \cup A$ and are also dependent on $S \cup A$.

A set map is *isotone* provided $\varphi(A) \subseteq \varphi(B)$ whenever $A \subseteq B \subseteq X$. A set map is *expansive* provided $A \subseteq \varphi(A)$ for all $A \subseteq X$. A set map over a finite set X that is isotone and expansive is *dolmatic*¹. In general, the first step in computing the φ -closure is to produce the dolmatic extension $\varphi_*(A)$ of φ :

$$\varphi_*(A) := A \cup \bigcup \{\varphi(S) : S \subseteq A\}.$$

In the case of Bingo closure the dependency map φ , as defined above, is isotone already so we can get its dolmatic extension as $\varphi_*(A) := A \cup \varphi(A)$. The importance of having a dolmatic function is that its iterates are always increasing and hence, in the finite case, eventually stabilize at the closure. Here a set is *closed* iff it contains all points dependent on it, and the *closure* of a set S is the smallest closed set containing S . These ideas are discussed in a rather different setting in [3] and are treated in abstract generality in [1, 2]. In this paper, we answer the question for Bingo closure:

What is the maximum number of times the dolmatic map must be applied to obtain the closure of a set?

¹From the Turkish verb “dolmak” which means “to fill”.

3	•	•	•	•	•	•	4	•
•	7	•	•	8	•	•	•	•
•	•	10	•	9	•	•	•	•
•	•	17	•	•	16	•	•	•
•	•	18	•	•	17	•	•	•
•	•	•	13	•	•	12	•	•
•	•	15	14	•	•	11	•	•
•	6	•	•	•	•	•	5	•
2	•	•	•	•	•	•	•	1

Figure 2: A 9×9 set of maximum depth

2 Maximum Depth for the Bingo Closure

Throughout the rest of this paper X will denote set of n^2 squares on an $n \times n$ grid. The closure is the Bingo closure described above. The Bingo closure of a set S will be denoted by $\mathcal{C}(S)$. The *depth* of a set S is the number of iterations of the dolmatic dependency map φ_* required to obtain the closure $\mathcal{C}(S)$ of S . In other words, the *depth* of S is the smallest d such that $\varphi_*^{d+1}(S) = \varphi_*^d(S)$.

A set S *spans* X provided its closure is X — that is, $\mathcal{C}(S) = X$. The little lemma below helps to show that the maximum depth can be achieved only by a spanning set.

Lemma 2.1 *If K is closed but not all of X , then there are at least 4 points of X not in K and at least 4 lines not contained in K .*

Proof. Suppose p is not in K . Since K is closed, K must be missing another point on the row and a point on the column containing p . Say, r and c are also missing from K in the row and column of p . Since r is missing from K , there is another point q missing from its column. Clearly $q \neq c$ since p and r are different points in the same row and hence lie in different columns. Thus 4 points are missing. The rows through p and c are not in K as are the columns through p and r . These are 4 lines not contained in K . ■

Theorem 2.2 *a) The depth of any non-spanning set S on an $n \times n$ board is at most $2n - 2$.*

b) The depth of a spanning set S at most $2n$.

Proof. Let $K = \mathcal{C}(S)$ be the Bingo closure of S and suppose the depth of S is λ . Each iteration of the dependency map completes at least one line

3	•	•	•	•	•	•	•	4	•
•	7	•	•	•	8	•	•	•	•
•	•	10	•	•	9	•	•	•	•
•	•	19	•	•	•	18	•	•	•
•	•	20	•	•	•	19	•	•	•
•	•	•	14	15	•	•	•	•	•
•	•	•	13	•	•	•	12	•	•
•	•	17	•	16	•	•	11	•	•
•	6	•	•	•	•	•	•	5	•
2	•	•	•	•	•	•	•	•	1

Figure 3: A 10×10 set of maximum depth

in K . Obviously, once a line is completed, it cannot be completed again. Thus the number of steps required to obtain the closure K is bounded by the number λ of lines contained in K . That is, $d \leq \lambda$. The number of lines in X is $2n + 2$, so if S is not spanning, the number of lines in K is at most $2n - 2$. Therefore (a) is established, so we can assume S is spanning and $K = X$.

Now consider the last element z added in forming the closure $\mathcal{C}(S) = X$ of S . If z is to be last, then it must complete all lines in X through z . There are at least two such lines (a row and a column) that are not complete before z is added. Thus two lines are used in the same iteration, giving an upper bound of $2n + 1$.

Now, in this case the last element to be added to S comes from two lines simultaneously. Hence two elements of S must also be added to S in the penultimate iteration. This also uses two lines in the same iteration, and so the upper bound becomes $2n$. ■

Board 2 of Figure 1 shows a spanning set of depth 10 in the traditional Bingo board with $n = 5$. The spanning set S consists of those squares marked with black dots. The numbers in the other squares indicate the order in which they are picked up by the dolmatic dependency map in forming the closure.

Theorem 2.3 *For $n \geq 5$, the Bingo board of side n has a spanning set S of depth at least $2n$.*

Proof. The proof is by construction of such a set of depth $2n$ for each $n \geq 5$. We proceed by induction, constructing the larger cases from smaller ones. We must split into the cases of even and odd n . The base case for the odd construction is given by Board 2 in Figure 1. Larger boards can

4	4	2
4	3	•
•	•	1

Board 3

•	1	•	2
6	•	5	•
5	•	4	•
4	•	•	3

Board 4

2	•	•	1	•	•
11	•	•	•	10	•
12	•	•	•	11	•
•	6	7	•	•	•
•	5	•	•	•	4
9	•	8	•	•	3

Board S

Figure 4: Small cases

be obtained by spiraling around the base case as shown in Figure 2. The base case is in the 5×5 rectangle enclosed by double lines. The numbers inside the base case have been increased by 8 since those squares appear 8 steps later in the closure process. The 4 new rows and 4 new columns get used first.

The situation is analogous in the even case, with the base case given in Board S of Figure 4. Figure 3 shows the construction for $n = 10$. Again the base case is in the 6×6 rectangle enclosed by double lines. For larger grids the same spiraling technique can be used. ■

For $n < 5$, the maximum depth is less than $2n$. For $n = 1$ and $n = 2$, the maximum depth is clearly just one. For $n = 3$, the maximum depth is 4. This is given by the construction below in Board 3 of Figure 4. This could be seen by merely enumerating all possible 3×3 boards. Instead however, let us consider again the last piece filled in.

The last four squares to be filled is significant and is the primary technique used in this paper in analyzing the maximum depth of small grids. The proof technique of Theorem 2.2 shows the fact that these last four squares filled in form the corners of a rectangle. Hence they will be called the *final rectangle*.

The comments above show that the last piece filled in comes from a rectangle of four, and that this rectangle must have depth at most three. Now to have depth three there must be a diagonal line which is on precisely one of the squares on the rectangle of four. No such rectangle exists on a 3×3 board, so the depth of the final rectangle is at most 2. Hence because each rectangle lies on both diagonals, there are at least 4 wasted lines. Hence the maximum depth is indeed $2 \cdot 3 + 2 - 4 = 4$.

For $n = 4$ the analysis is a little more tricky. This is because of the necessary overlap of the diagonals and the final rectangle. There are several cases to worry about though.

If a rectangle is left empty four squares are left empty and there are least 5 (two horizontal, two vertical, and one diagonal) wasted lines giving

a maximum depth of $2 \cdot 4 + 2 - 5 = 5$. If on the other hand we are to generate all 16 tiles, then the final rectangle must overlap either both diagonals. or a single diagonal twice. If it has a single diagonal twice, one cannot come onto a final rectangle via a diagonal and thus at all. This leaves only the case that it contains both diagonals. Hence one of the squares must be filled from two lines - one diagonal and one from horizontal or vertical. This gives an additional two wasted lines: the diagonal, and whatever line filled the square that used the diagonal. This gives a maximum depth of $2 \cdot 4 + 2 - 2 - 1 - 1 = 6$. Such a construction is given below in Board 4 of Figure 4.

References

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